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CALCULATION OF ELECTROMAGNETIC SCATTERING BY A PERFECT CONDUCTO--ETC(U)

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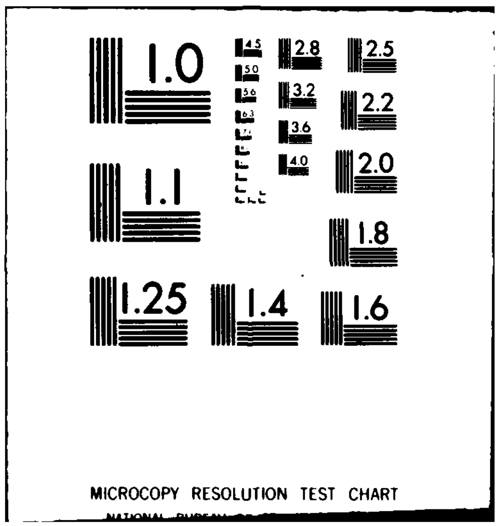
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CALCULATION OF ELECTROMAGNETIC SCATTERING BY A PERFECT CONDUCTOR

BY A. K. AZIZ M. R. DORR R. B. KELLOGG

RESEARCH AND TECHNOLOGY DEPARTMENT

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An electromagnetic wave of given shape and frequency falls on a perfectly conducting body or collection of bodies. It is desired to calculate the resulting field. A new method is presented for the numerical solution of this problem, a computer program implementing the method is described, and the results of some numerical calculations are given.

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SUMMARY

A new method is presented for calculating the scattering of an arbitrary electromagnetic wave by a bounded, perfectly conducting body of general shape. The strategy is to replace the corresponding exterior boundary-value problem by an approximate problem on the boundary of the scattering body. This involves the introduction of a certain bilinear form and non-local boundary operator, together with the use of a special class of known solutions of the reduced Maxwell's equations satisfying the Sommerfeld radiation boundary conditions at infinity. Two computer programs implementing this method are described and numerical results showing the successful application of this method to some model problems are presented.

B. F. DESAVAGE
By direction

CONTENTS

1. INTRODUCTION

The numerical calculation of scattering by an arbitrarily shaped perfect conductor or dielectric body has received much attention in recent years. The problem is important in many areas including the design of waveguides and missile fuzes, the assessment of damage by an electromagnetic pulse, and the study of the biological effects of microwave radiation. Mathematically, the problem has the form of an exterior boundary value problem for a system of elliptic partial differential equations. The problem has particular difficulties arising from the large number of unknowns, the unbounded domain, and the fact that the solution is oscillatory for large values of frequency.

The principal techniques for the numerical solution of scattering problems are the method of integral equations, and the finite difference or finite element method. We discuss briefly these methods.

In the method of integral equations, one starts with a formulation of the boundary value problem as an integral equation on the surface of the scattering body; the unknown may, for example, be the current density on the surface. An excellent survey of such formulations, in the case of the reduced wave equation, is given in [1]. The integral equation is solved numerically, either by using a quadrature formula or a Galerkin method to reduce the problem to a finite system of linear equations. The principal advantage of the method is that the dimension of the problem has been reduced; one must find a vector field on the two dimensional surface of the scattering body, instead of in the three dimensional exterior region. On the other hand, the system of linear equations has a full (non sparse) coefficient matrix. Also, the integral equation contains a weakly singular kernel that comes from the fundamental solution of the problem; the resulting surface integrals may be difficult to evaluate with sufficient accuracy. Examples of the integral equation approach are contained in [2,3].

With the second method, the boundary condition at infinity is replaced by a boundary condition on the surface of a large sphere containing the scattering body. This approach is used in [4]. The resulting boundary value problem is discretized by means of a finite difference or finite element method. This method has the advantage of producing a simple, sparse coefficient matrix. On the other hand, the order of the matrix will be larger, because the unknown is now a vector field in a three dimensional domain. A judicious use of

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1. Kleinman, R. E. and Roach, G. F., "Boundary Integral Equations for the Three Dimensional Helmholtz Equation," SIAM Rev., Vol. 16, pp. 214-236, 1974.
2. McDonald, B., Friedman, M., and Wexler, A., "Variational Solution of Integral Equations," IEEE Transactions on Microwave Theory and Technique, Vol. MTT-24, pp. 237-248, 1974.
3. Livesay, D. E. and Chen, K-M., "Electromagnetic Fields Induced Inside Arbitrarily Shaped Biological Bodies," Ibid., pp. 1273-1280.
4. Baylis, A., Gunzburger, M., and Turkel, E., "Boundary Conditions for the Numerical Solution of Elliptic Equations in Exterior Regions," Report 80-1, ICASE, NASA-Langley, Hampton, Va., 1980.

graded meshes at a large distance from the scattering body will help alleviate this problem [5]. Examples of the use of finite differences or finite elements are contained in [6,7,8]. The finite element method is especially appropriate for the problem of penetration of electromagnetic fields into an inhomogeneous, absorbing body. To treat such problems, one couples the finite element procedure inside the body with an integral equation or other technique on the surface of the absorbing body. For some work on this, see [9,10,11]. A particularly successful coupling method has been developed by Waterman [12,13]. In Waterman's method, an integral equation is used on the boundary, and the external fields are expanded in a series of harmonic vector fields. This eliminates the singularity from the kernel of the equation and gives a representation of the approximate solution in terms of fields which already satisfy the differential equations of the problem. It should be noted, however, that although the resulting integrands no longer contain singularities, they contain oscillating terms, and care in their evaluation is still required. Examples of further work along this line are contained in [14,15,16].

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5. Goldstein, C., "Numerical Methods for Helmholtz Type Equations in Unbounded Domains," BNL-26543, Brookhaven Laboratory, Brookhaven, N.Y., 1979.
6. McDonald, B. H. and Wexler, A., "Finite Element Solution of Unbounded Field Problems," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-20, pp. 841-847, 1972.
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9. Zienkiewicz, O. C., Kelly, D. W., and Bettess, P., "The Coupling of the Finite Element Method and Boundary Solution Procedures," International J. for Numer. Methods in Engineering, Vol. 11, pp. 355-375, 1977.
10. Brezzi, F. and Johnson, C., "On the Coupling of Boundary Integral and Finite Element Methods," Report 77.15R, Dept. of Computer Science, Chalmers University of Technology, Göteborg, Sweden, 1977.
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12. Waterman, P. C., "Matrix Formulation of Electromagnetic Scattering," Proc. IEEE, Vol. 53, pp. 805-812, 1965.
13. Waterman, P. C., "New Formulation of Acoustic Scattering," J. of the Acoustical Soc., Vol. 45, pp. 1417-1429, 1969.
14. Barber, P. and Yeh, C., "Scattering of Electromagnetic Waves by Arbitrarily Shaped Dielectric Bodies," Applied Optics, Vol. 14, pp. 2864-2872, 1975.
15. Barber, P., "Resonance Electromagnetic Absorption by Nonspherical Dielectric Objects," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-25, pp. 373-381, 1977.
16. Morgan, M. A. and Mei, K. K., "Finite Element Computation of Scattering by Inhomogeneous Penetrable Bodies of Revolution," IEEE Transactions on Antennas and Propagation, Vol. AP-27, pp. 202-214, 1979.

The method proposed here has similarities with the method of Waterman, in that we also use harmonic vector fields. Our starting point is different, however, in that we view the problem as a system of differential equations coupled with a complicated non-local boundary condition. (See [17] for a similar point of view.) The method was first developed for the penetration problem [18] and was described in a preliminary way in [19] and [20].

2. MATHEMATICAL FORMULATION

Let Ω be a bounded, perfectly conducting, three dimensional domain with boundary Γ. We allow the possibility that Ω is a disconnected domain, in which case Γ is a collection of disjoint closed surfaces. Let Ω_0 be the set of points not in Ω. We are given an incident electromagnetic wave, $\underline{E}^0$, $\underline{H}^0$, of frequency ω. Thus, $\underline{E}^0$ and $\underline{H}^0$ are vector fields defined in all space, which satisfy the reduced Maxwell's equations

$$\underline{\nabla} \times \underline{E}^0 = \alpha\underline{H}^0,\ \underline{\nabla} \times \underline{H}^0 = \beta\underline{E}^0.$$

If the wave is propagating in a vacuum, $\alpha = i\omega\mu_0$, $\beta = -i\omega\varepsilon_0$, where $\varepsilon_0 = 8.8574 \times 10^{-12}$ meters/farad, $\mu_0 = 1.2566 \times 10^{-6}$ meters/henry. In general, $\alpha = i\alpha_0$, $\beta = -i\beta_0$, where α_0, β_0 are positive real numbers. Let $\underline{E}$, $\underline{H}$ be the electromagnetic wave resulting from the scattering of the incident wave by the perfectly conducting body Ω. Let $\underline{E}^1 = \underline{E} - \underline{E}^0$, $\underline{H}^1 = \underline{H} - \underline{H}^0$ be the scattered wave. Then $\underline{E}(x)$, $\underline{H}(x)$ are defined for $x\varepsilon\Omega_0$, and are determined by the following set of equations:

(2.1a)	$\underline{\nabla} \times \underline{E} = \alpha\underline{H}$	$x\varepsilon\Omega_0$
(2.1b)	$\underline{\nabla} \times \underline{H} = \beta\underline{E}$	$x\varepsilon\Omega_0$
(2.2)	$\underline{E} \times \underline{n} = 0,$	$x\varepsilon\Gamma$
(2.3a)	$\underline{E}^1, \underline{H}^1 = O(r^{-1}),$	$r\to\infty$
(2.3b)	$\underline{e}_r \times \underline{E}^1 - \sqrt{\alpha_0/\beta_0}\ \underline{H}^1 = o(r^{-1}),$	$r\to\infty$
(2.3c)	$\underline{e}_r \times \underline{H}^1 + \sqrt{\beta_0/\alpha_0}\ \underline{E}^1 = o(r^{-1}),$	$r\to\infty$

17. MacCamy, R. C., "Variational Procedures for a Class of Exterior Interface Problems," Report 79-9, Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pa., 1979.
18. Aziz, A. K. and Kellogg, R. B., "Finite Element Analysis of a Scattering Problem," Report BN-917, Inst. for Phys. Sci. and Tech., Univ. of Md., College Park, Md., 1979.
19. Aziz, A. K. and Kellogg, R. B., "A Scattering Problem for the Helmholtz Equation," in <u>Advances in Computer Methods for Partial Differential Equations-III</u>, R. Vichnevetsky, editor, IMACS, 1979.
20. Kellogg, R. B., "A Scattering Problem for Maxwell's Equations," Ibid.

Here $\underline{n}$ denotes the outward pointing unit normal to Γ, $r = |x|$, and $\underline{e}_r$ denotes the unit vector in the radial direction.

The system (2.1a,b) comprises a system of six partial differential equations in the six unknowns consisting of the components of $\underline{E}$ and $\underline{H}$. The equations (2.1), (2.2), (2.3) form an exterior boundary value problem for this system; (2.2) is the boundary condition on Γ, and (2.3) are the boundary conditions at infinity. This problem has been treated, for example, in [21], where it is shown that, for reasonable surfaces Γ and incident waves $\underline{E}^0$, $\underline{H}^0$, the problem has a unique solution.

It is convenient to discuss the boundary value problem in terms of the scattered wave. For this, let $\underline{f}$ be a tangential vector field on Γ, and consider the boundary value problem defined by

(2.4a) $\quad \underline{\nabla} \times \underline{E}^1 = \alpha \underline{H}^1, \qquad x \varepsilon \Omega_0,$

(2.4b) $\quad \underline{\nabla} \times \underline{H}^1 = \beta \underline{E}^1, \qquad x \varepsilon \Omega_0,$

(2.5) $\quad \underline{n} \times \underline{E}^1 = \underline{f}, \qquad x \varepsilon \Gamma,$

and (2.3). From [21], it is known that this problem has a unique solution. If $\underline{f} = -\underline{n} \times \underline{E}^0$, then the solution $\underline{E}^1$, $\underline{H}^1$ of (2.4), (2.5), (2.3) gives the scattered wave for the original problem. Thus, it suffices to solve (2.4), (2.5), (2.3).

Let $\underline{f}$ be a tangential vector field on Γ. We define another tangential field, $K\underline{f}$, as follows. Let $\underline{E}^1$, $\underline{H}^1$ be the solution of (2.4), (2.5), (2.3), and let $K\underline{f} = \underline{n} \times \underline{H}^1$. The operator K maps tangential vector fields into tangential vector fields. We also define a bilinear form, B, on tangential vector fields. If $\underline{f}$ and $\underline{g}$ are two tangential vector fields on Γ, we define

$$B(\underline{f}, \underline{g}) = \oint_\Gamma \underline{n} \cdot \underline{f} \times K\underline{g} \, d\Gamma \ .$$

The operator K and the bilinear form B are used in the formulation of our numerical method. We prove that the bilinear form is symmetric.

<u>Lemma 1</u>. $B(\underline{f}, \underline{g}) = B(\underline{g}, \underline{f})$.

Proof. Let $\underline{E}^1$, $\underline{H}^1$ be the solution of (2.4), (2.5), (2.3), and let $\underline{E}^2$, $\underline{H}^2$ be the solution of (2.4), (2.5), (2.3) with $\underline{f}$ replaced by $\underline{g}$. Let B_r be a ball of radius r with boundary S_r. Let r be chosen so large that $\Omega \subset B_r$, and let $\Omega_{0,r} = \Omega_0 \cap B_r$. We have

$$\beta \underline{E}^1 \cdot \underline{E}^2 - \alpha \underline{H}^1 \cdot \underline{H}^2 = \underline{\nabla} \cdot (\underline{H}^1 \times \underline{E}^2).$$

21. Müller, C., <u>Foundations of the Mathematical Theory of Electromagnetic Waves</u>, Springer-Verlag, New York, 1969.

Integrating this over $\Omega_{0,r}$, we obtain

$$-\oint_\Gamma \underline{n}\cdot\underline{H}^1 \times \underline{E}^2 d\Gamma + \oint_{S_r} \underline{e}_r\cdot\underline{H}^1 \times \underline{E}^2 d\Gamma = \int_{\Omega_{0,r}} [\beta\underline{E}^1\cdot\underline{E}^2 - \alpha\underline{H}^1\cdot\underline{H}^2]dx.$$

The first term on the left side is

$$\oint_\Gamma \underline{n}\cdot\underline{E}^2 \times \underline{H}^1 d\Gamma = \oint_\Gamma \underline{n}\cdot\underline{g} \times K\underline{f}d\Gamma$$
$$= B(\underline{g}, \underline{f}).$$

Using (2.3), we find that the second term on the left side is

$$\oint_{S_r} \underline{E}^2\cdot\underline{e}_r \times \underline{H}^1 d\Gamma = -\sqrt{\beta_0/\alpha_0} \oint_{S_r} \underline{E}^2\cdot\underline{E}^1 d\Gamma + \delta(r),$$

where $\delta(r) \to 0$ as $r \to \infty$. Since $B(\underline{g}, \underline{f})$ is independent of r, it follows that the expression

$$\int_{\Omega_{0,r}} (\beta\underline{E}^1\cdot\underline{E}^2 - \alpha\underline{H}^1\cdot\underline{H}^2)dx + \sqrt{\beta_0/\alpha_0} \oint_{S_r} \underline{E}^1\cdot\underline{E}^2 d\Gamma$$

has a finite limit as $r \to \infty$, and that

$$B(\underline{g}, \underline{f}) = \lim_{r\to\infty} \left\{ \int_{\Omega_{0,r}} (\beta\underline{E}^1\cdot\underline{E}^2 - \alpha\underline{H}^1\cdot\underline{H}^2)dx + \sqrt{\beta_0/\alpha_0} \oint_{S_r} \underline{E}^1\cdot\underline{E}^2 d\Gamma \right\}. \tag{2.6}$$

Since the right side of (2.6) is unchanged if $\underline{E}^1$, $\underline{H}^1$ and $\underline{E}^2$, $\underline{H}^2$ are interchanged, the left side of (2.6) is unchanged if $\underline{f}$ and $\underline{g}$ are interchanged, and the lemma is proved.

We now show that the bilinear form is definite. This will be used in the next section to show that the finite system of equations which our method produces always has a solution. To state the result, we let $\bar{z}$ denote the complex conjugate of a complex number z.

Lemma 2. If $B(\underline{f}, \overline{\underline{f}}) = 0$, then $\underline{f} = 0$.

Proof. Set $\underline{g} = \overline{\underline{f}}$ in (2.6), and note that α and β are imaginary. Then

$$\mathrm{Re}B(\underline{f}, \overline{\underline{f}}) = (\beta_0/\alpha_0)^{1/2} \lim_{r\to\infty} \oint_{S_r} |\underline{E}^1|^2 d\Gamma. \tag{2.7}$$

In particular, we find that the limit on the right side of (2.7) exists. Suppose $B(\underline{f}, \overline{\underline{f}}) = 0$. Then

$$\lim_{r\to\infty} \oint_{S_r} |\underline{E}^1|^2 d\Gamma = 0.$$

It follows from a theorem of Rellich [21, Theorem 15] that $\underline{E}^1 \equiv 0$. Hence $\underline{f} = 0$.

3. THE APPROXIMATION SCHEME

We describe our numerical method in terms of the scattered wave $\underline{E}^1$. Let $\underline{g}$ be any tangential vector field on Γ. Then since $\underline{n} \times \underline{E}^1 = -\underline{n} \times \underline{E}^0$ on Γ, we have

$$\begin{aligned} \underline{n}\cdot\underline{E}^1 \times K\underline{g} &= \underline{n} \times \underline{E}^1\cdot K\underline{g} \\ &= -\underline{n} \times \underline{E}^0\cdot K\underline{g} \\ &= -\underline{n}\cdot\underline{E}^0 \times K\underline{g}. \end{aligned}$$

Integrating this over Γ, and letting $\underline{f}$ denote the tangential projection of the restriction of $\underline{E}^1$ to Γ, we obtain

$$B(\underline{f}, \underline{g}) = -\oint_{\Gamma} \underline{n}\cdot\underline{E}^0 \times K\underline{g}d\Gamma.$$

This equation is the basis for our numerical method. We pick a finite dimensional collection S of tangential vector fields on Γ, and we define our approximate solution $\underline{f}\varepsilon S$ by

$$B(\underline{f}, \underline{g}) = -\oint_{\Gamma} \underline{n}\cdot\underline{E}^0 \times K\underline{g}d\Gamma, \quad \underline{g}\varepsilon S. \tag{3.1}$$

The system (3.1) gives rise to a finite system of linear equations whose solution determines the vector field $\underline{f}\varepsilon S$. This approximation scheme seems to suffer from two defects. It is not clear how to obtain the approximate scattered field in Ω_0 from $\underline{f}$, and it is not clear how to evaluate the operator K which appears in (3.1). These defects are overcome by a judicious choice of subspace, which we now describe.

Let $x^*\varepsilon\Omega$ be given, and let (r,θ,ϕ) be a system of spherical coordinates with the origin at x^*. In Stratton [22, p. 416], there are constructed a family of vector fields,

$$\underline{m}_{\substack{e\\o}mn},\ \underline{n}_{\substack{e\\o}mn},\ m = 0,1,\cdots,n,\ n = 1,2,\cdots. \tag{3.2}$$

which satisfy the following properties.

(i) The fields are regular for $x \neq x^*$, and hence are regular in Ω_0.

(ii) The fields satisfy

$$\underline{\nabla} \times \underline{m}_{\substack{e\\o}mn} = \sqrt{\alpha_0\beta_0}\ \underline{n}_{\substack{e\\o}mn},$$

22. Stratton, J. A., Electromagnetic Theory, McGraw-Hill, New York, 1941.

$$\underline{\nabla} \times \underline{n}_{\substack{e\\o}mn} = \sqrt{\alpha_0 \beta_0}\, \underline{m}_{\substack{e\\o}mn} .$$

(iii) The fields satisfy (2.3).

Note that we must take the Hankel functions $z_n(\rho) = h_n^1(\rho)$ in Stratton's formulas to satisfy (iii).

We fix an integer $N > 0$, and let C_N denote the collection of vector fields (3.2) for $0 \le m \le n$, $1 \le n \le N$. Let S_N denote the collection of tangential vector fields on Γ which are of the form $\underline{n} \times (\underline{U}|_\Gamma)$, where $\underline{U} \epsilon C_N$. There are $2N(N+2)$ linearly independent fields in S_N. If $\underline{g} \epsilon S_N$, using (ii), we may easily calculate $K\underline{g}$. If $\underline{f} \epsilon S_N$ is the solution of (3.1), then $\underline{f}$ comes from a vector field $\underline{U}$ in C_N; the field $\underline{U}$ is the desired approximate scattered field, and may be easily evaluated at points of Ω_0. We have therefore shown how to overcome the defects of using (3.1).

We now discuss the system of equations arising from the use of (3.1). We arrange the fields (3.2) of C_N in a definite order and denote them by $\underline{F}_\mu$, $1 \le \mu \le M = 2N(N+2)$. We let $\underline{f}_\mu = \underline{n} \times (\underline{F}_\mu|\Gamma)$. Thus, the $\underline{f}_\mu$, $1 \le \mu \le M$, are tangential vector fields on Γ which form a basis for S_N. Writing the desired solution $\underline{f}$ of (3.1) as $\underline{f} = \Sigma c_\mu \underline{f}_\mu$, we obtain the linear system

$$(3.3) \qquad \sum_{\nu=1}^{M} c_\nu B(\underline{f}_\mu, \underline{f}_\nu) = - \oint_\Gamma \underline{n} \cdot \underline{E}^0 \times K\underline{f}_\mu d\Gamma, \; 1 \le \mu \le M.$$

We set $a_{\mu\nu} = B(\underline{f}_\mu, \underline{f}_\nu)$, and we let $A = [a_{\mu\nu}]$ denote the coefficient matrix. We have the following

Theorem. The complex matrix A is symmetric, nonsingular, has nonsingular principal minors, and admits a Cholesky factorization $A = LL^T$.

Proof. The symmetry of A follows from Lemma 1. The nonsingularity of A, and of the principal minors of A, follows from Lemma 2. The existence of the Cholesky decomposition then follows from the arguments of [23, Thm. 3.1]. (Note, however, that A is complex and symmetric, not Hermitian.)

Remark 1. The right side of (3.1) could be written in terms of the bilinear form. However, an application of Lemma 1 would not enable us to express the right side in terms of the incident magnetic field. This is because $K\underline{E}^0 \ne \underline{H}^0$, since the incident field does not satisfy (2.3). To avoid this possible confusion, we have chosen to express the right side of (3.1) as we have done.

Remark 2. An error analysis for the method proposed here will appear in a forthcoming paper.

23. Stewart, G. W., Introduction to Matrix Computations, Academic Press, New York, 1973.

4. COMPUTER IMPLEMENTATION

Two computer programs, PCISH (Perfect Conduction In Spherical Harmonics) and PSYM (an axisymmetric version of PCISH), have been developed to implement the numerical scheme described in §3. PCISH is the more general of the two since it is capable of computing the electric (or magnetic) field resulting from the scattering of an arbitrary electromagnetic wave $\underline{E}^0$ by a simply-connected, infinitely conducting body Ω of arbitrary shape. PSYM is an off-shoot of PCISH which is designed to handle the specific class of problems in which the boundary of Ω is a surface of revolution and the incident wave $\underline{E}^0$ is a plane wave propagating along the axis of symmetry.

In PCISH, it is assumed that the boundary of Ω can be suitably approximated by a closed surface which is the union of a number of quadrilaterals. The vertices of the quadrilaterals and information giving the assignments of vertices to quadrilaterals comprise a major portion of the input. Other input parameters are the frequency, orientation, and shape of the incident wave, the center and maximum order N of the subspace C_N, the quadrature order, and the coordinates of the points in space at which the scattered field is to be calculated. The input to PSYM is similar (except for specifying the incident wave) but is much simpler since the user need only supply a suitable number of points lying on a curve Γ which generates the boundary of Ω by rotation about an axis of symmetry. The actual rotation of Γ is carried out implicitly by the program.

Most of the computations performed by PCISH and PSYM center around the evaluation of the coefficient matrix and the right-hand side of the linear system (3.3). In view of the Theorem, both PCISH and PSYM use Gauss elimination without pivoting to solve the matrix equation.

As for the evaluation of the coefficient matrix and right-hand side, PCISH performs a two dimensional quadrature (based on the trapezoidal rule) over each quadrilateral of the surface representing the boundary of Ω. PSYM, on the other hand, performs only a one-dimensional quadrature (also based on the trapezoidal rule) over a piecewise linear arc approximating Γ, which is formed by joining the given input points on Γ by line segments. The reason that only a one dimensional quadrature is necessary is that in the case in which $\partial\Omega$ is a surface of revolution, the quadrature in the ϕ direction has been carried out exactly by hand (requiring only the integration of some trigonometric polynomials) and the corresponding formulas placed in the program. In both PCISH and PSYM, the quadrature order is an input parameter.

The assumption that the boundary of Ω is a surface of revolution has another large advantage in that, due to certain symmetries which exist in this case, many of the matrix entries are zero. Moreover, if the incident wave $\underline{E}^0$ is assumed to be a plane wave propagating along the axis of symmetry, certain of the right-hand side entries also become zero, and the originally $2N(N+2)$-dimensional system reduces to a $2N$-dimensional system where N is again the maximum order of the subspace C_N. Instead of using all of the vector fields (3.2), we may take subspaces generated only by the vector fields

$$\underline{m}_{o1n},\ \underline{n}_{e1n},\ n=1,2,\cdots,N.$$

The advantage of such a reduction is obvious, especially for large values of N.

In order to evaluate the integrand in the bilinear form at each quadrature point, one needs to be able to calculate the Hankel functions h_n^1 and the associated Legendre polynomials P_n^m used in the definition of the fields of (3.2). The routines used by PCISH and PSYM are as follows.

The Hankel functions $h_n^1 = j_n + iy_n$ are computed at a given point $\rho > 0$ using a backward recursion formula to calculate j_n, the nth order spherical Bessel function of the first kind, and a forward recursion formula to calculate y_n, the nth order spherical Bessel function of the second kind. Starting with $x_{M-1} = 1$ and $x_M = 0$ for M sufficiently large, we recursively compute x_n, $n = 0,1,\cdots,M-2$ by

$$x_n = (2n + 3)\,\frac{x_{n+1}}{\rho} - x_{n+2}\ .$$

We then set

$$j_n(\rho) = \frac{\sin\rho}{\rho x_0}\, x_n .$$

As for the y_n, we set $y_0(\rho) = -\frac{1}{\rho}\cos\rho$, $y_1(\rho) = -\frac{1}{\rho^2}\cos\rho - \frac{1}{\rho}\sin\rho$ and compute

$$y_n(\rho) = \frac{2n-1}{\rho}\, y_{n-1}(\rho) - y_{n-2}(\rho)\ .$$

We then set $h_n^1(\rho) = j_n(\rho) + iy_n(\rho)$. One also needs certain derivatives of the h_n^1 and for these, we employ the following useful formula (see 10.1.21 of [24])

$$\frac{1}{\rho}(\rho h_n^1)' = h_{n-1}^1 - \frac{n}{\rho}\, h_n^1 \qquad (' = \frac{d}{d\rho})\ .$$

To calculate the associated Legendre polynomials P_n^m of degree n, order m, at a given point $\eta, -1 \leq \eta \leq 1$ we first observe that using the Rodriguez formula for $P_n = P_n^0$, we have

$$\begin{aligned} P_n^m(\eta) &\overset{\text{def}}{=} (1 - \eta^2)^{\frac{m}{2}}\,\frac{d^m P_n(\eta)}{d\eta^m} \\ &= (1 - \eta^2)^{\frac{m}{2}}\,\frac{d^m}{d\eta^m}\left[\frac{1}{2^n n!}\,\frac{d^n}{d\eta^n}\,(\eta^2 - 1)^n\right] \\ &= \frac{(1 - \eta^2)^{\frac{m}{2}}}{2^n n!}\,\frac{d^{m+n}}{d\eta^{m+n}}\,(\eta^2 - 1)^n\ . \end{aligned}$$

24. Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, U.S. Government Printing Office, 1964.

Therefore,

$$P_m^m(\eta) = \frac{(1-\eta^2)^{\frac{m}{2}}}{2^m m!}\frac{d^{2m}}{d\eta^{2m}}(\eta^2-1)^m$$

$$= \frac{(1-\eta^2)^{\frac{m}{2}}(2m)!}{2^m m!}$$

$$= \frac{(2m)(2m-1)}{2m}\frac{[2(m-1)]!}{2^{m-1}(m-1)!}(1-\eta^2)^{\frac{m-1}{2}}\sqrt{1-\eta^2}$$

$$= (2m-1)\sqrt{1-\eta^2}\,P_{m-1}^{m-1}(\eta) \quad .$$

Since $P_0^0(\eta) \equiv 1$, we may recursively calculate $P_m^m(\eta)$ for any m. To obtain $P_n^m(\eta)$ for $m \neq n$, we observe that $P_n^m(\eta) \equiv 0$ for $n < m$ and apply the recursion formula (see 8.5.3 of [24])

$$(n - m+1)P_{n+1}^m(\eta) = (2n+1)\eta P_n^m(\eta) - (n+m)P_{n-1}^m(\eta) \quad .$$

We also need the derivatives $\frac{d}{d\eta}P_n^m$. Using the fact that $\frac{d}{d\eta}P_0^m(\eta) \equiv 0$ for all m, we apply the formula (see 8.5.4 of [24])

$$(1-\eta^2)\frac{dP_n^m}{d\eta} = (n+m)P_{n-1}^m - n\eta P_n^m$$

for $n \geq 1$, $m \geq 0$.

Once the linear system has been solved by Gauss elimination, the approximate scattered field $\underline{E}^1$ is assembled and may be evaluated at any given point x in space. The relevant quantity for many applications is the <u>radar cross-section</u> (RCS) defined by

$$\sigma(x) = \frac{4\pi R^2|\underline{E}^1(x)|^2}{|\underline{E}^0(x)|^2} \tag{4.1}$$

where $\underline{E}^0$ is the incident wave and R is the distance of the point x from the body Ω. To calculate the scattered field at infinity (i.e. the farfield) one takes the limit of (4.1) as $R\to\infty$. Computationally, this is achieved by using an asymptotic form of the Hankel functions when the scattered field $\underline{E}^1$ is assembled from the solution of the linear system.

5. NUMERICAL RESULTS

Both PCISH and PSYM have been developed and executed on the CDC 6500 computer located at NSWC/White Oak. These programs have been applied to a number of relatively simple model problems in order to test the program

capabilities and to compare computed results with results found in the literature. Since all of the problems attempted to date have been axially symmetric, the more specialized program PSYM was used to obtain the numerical results presented below.

As mentioned in §4, the maximum order N and center x* of the subspace C_N and the quadrature order are input parameters. In order to verify our results for a given body Ω, we manipulate these quantities in the following way. We first determine the proper value for N by successively increasing this parameter until the RCS stabilizes to some pre-determined number of significant digits. However, since an increase in N causes an increase in the oscillation of the associated Legendre polynomials used in their definition, we must simultaneously increase the quadrature order to insure an accurate evaluation of the matrix system (3.3). For a given value of N, the quadrature order is increased until the RCS again stabilizes to a pre-determined tolerance. Once these quantities have been determined, we perform a consistency check by varying the subspace center x*. Since this should in theory have no effect on the farfield profile, this seems to be a fairly rigorous test of the computed values. Obviously, for each new body Ω this process involves a fair number of program runs, but it is hoped that a further investigation of our method will lead to a better understanding of the interplay among these various quantities and their relationship to the other relevant parameters of the problem.

In the case in which Ω is an infinitely conducting sphere, an exact solution is known and can be found in many classical texts on electromagnetic theory (see e.g. [22]). PSYM produced extremely good farfield backscattering results in this case, even well into the resonance region, as can be seen by comparing Fig. 1 with a similar plot obtained from the exact solution on p. 148 of [25].

The case in which Ω is a right circular cone is also studied in the literature [25]. For our tests, we used a right circular cone circumscribed about a 1 m. sphere with a 15° half-angle at the vertex. The plane wave $\underline{E}^0$ was incident upon the vertex and propagated along the axis of symmetry. Fig. 2 shows the farfield backscattering results produced by PSYM in the Rayleigh region and extending a short way into the resonance region. These results seem to agree with other computed and experimental results reported in [25] (in particular, see the figure on p. 392 of [25]).

For some applications, it is desired to calculate the scattered field close to the body Ω. This presents no difficulties for PSYM, and Fig. 3 displays some partial results in this direction for the same conical scatterer as described above. For each of the frequencies 1MHz, 10MHz, and 50MHz (corresponding to $\frac{2\pi a}{\lambda}$ = .027, .273, and 1.365 respectively in Fig. 2), the scattered field was evaluated on circles of radii 4m., 10m., and 50m. lying in the plane parallel to $\underline{E}^0$ which contains the axis of symmetry of the cone. In order to obtain a farfield profile, calculations were also made on a fourth circle, concentric with the other

25. Ruck, G. T., editor, Radar Cross Section Handbook, Vol. 1, Plenum Press, New York, 1970.

three circles, which had an effectively infinite radius. The frequency 1 MHz is in the Rayleigh region and yields a constant RCS in the farfield. The frequency 50 MHz is in the resonance region, and the frequency 10 MHz is on the border between the Rayleigh and resonance regions (see [26]).

26. Crispin, J. W., Jr., and Maffett, A. L., "Radar Cross-Section Estimation for Simple Shapes," Proc. IEEE, Vol. 53, 833-847, 1965.

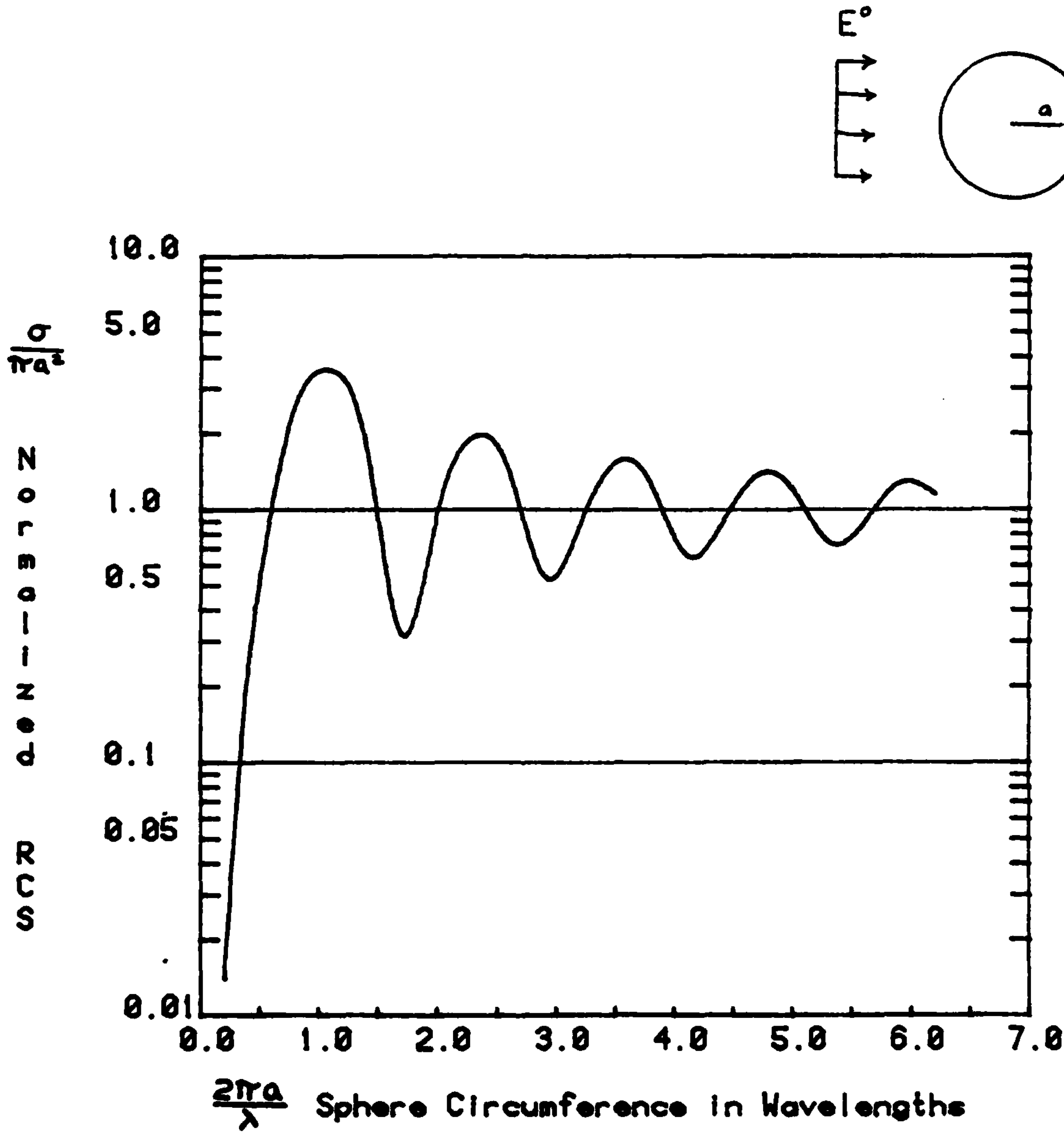


FIGURE 1 BACKSCATTERING OF A PLANE WAVE BY A PERFECTLY CONDUCTING SPHERE OF RADIUS a

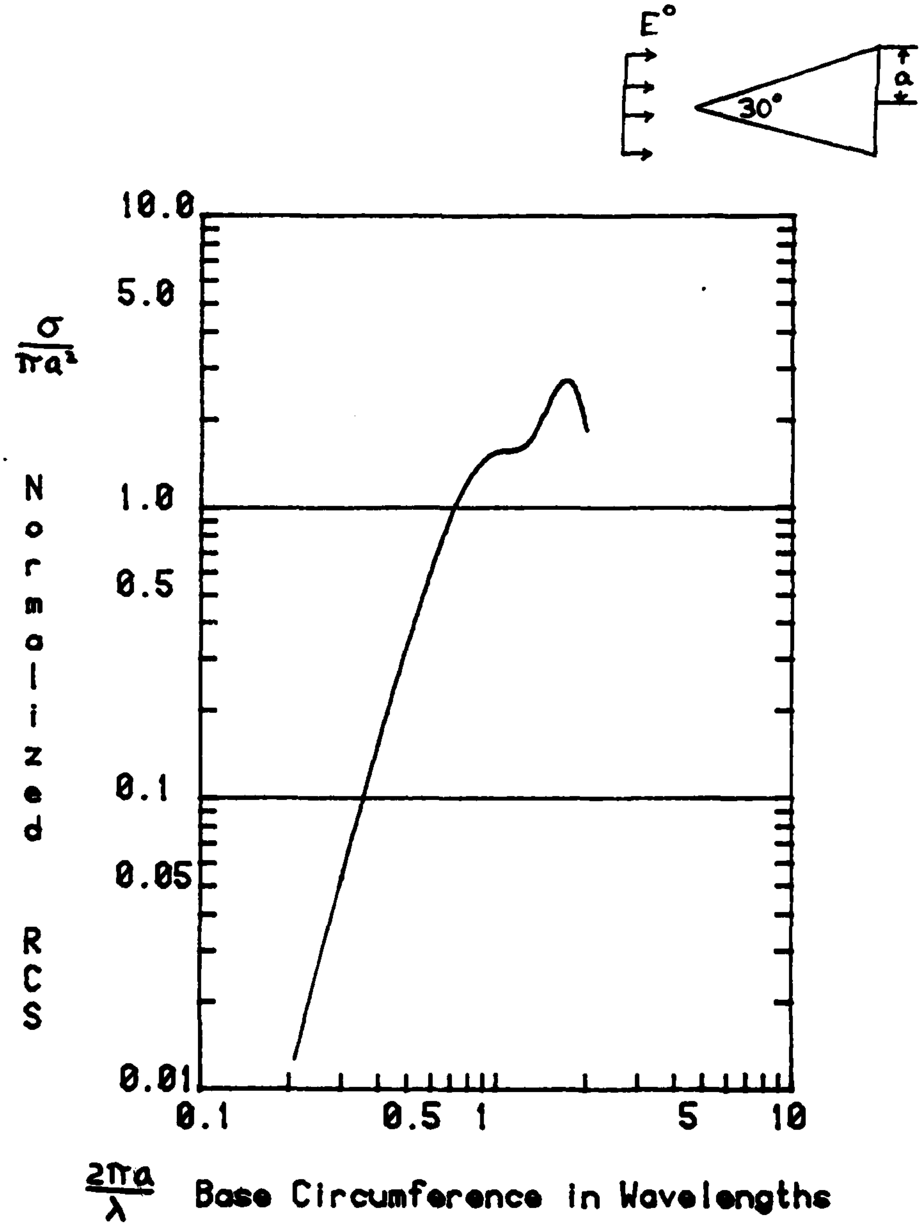
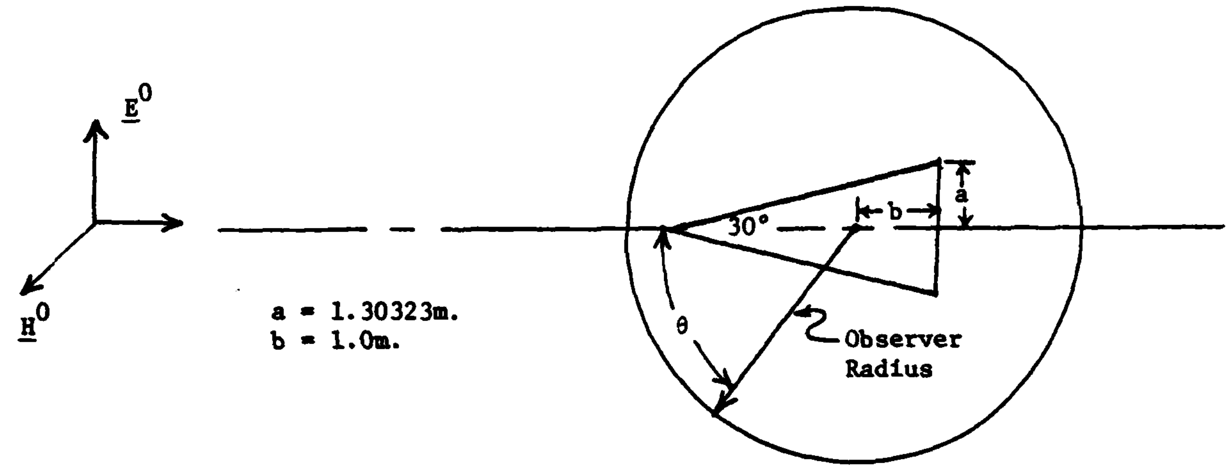


FIGURE 2 BACKSCATTERING OF A PLANE WAVE BY A RIGHT CIRCULAR CONE

Nearfield Results for a Cone



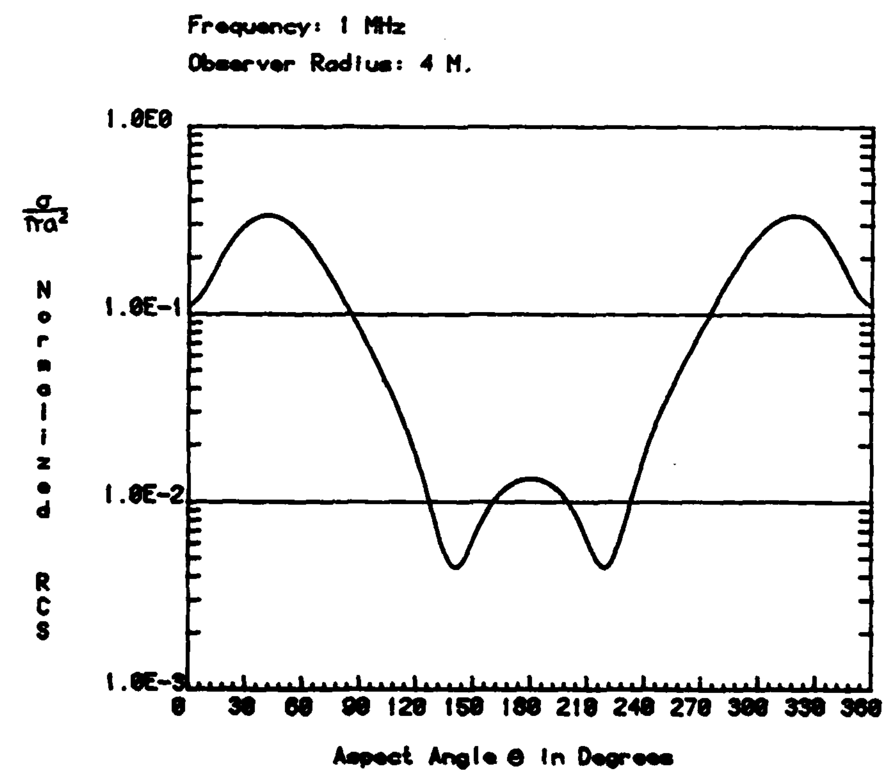
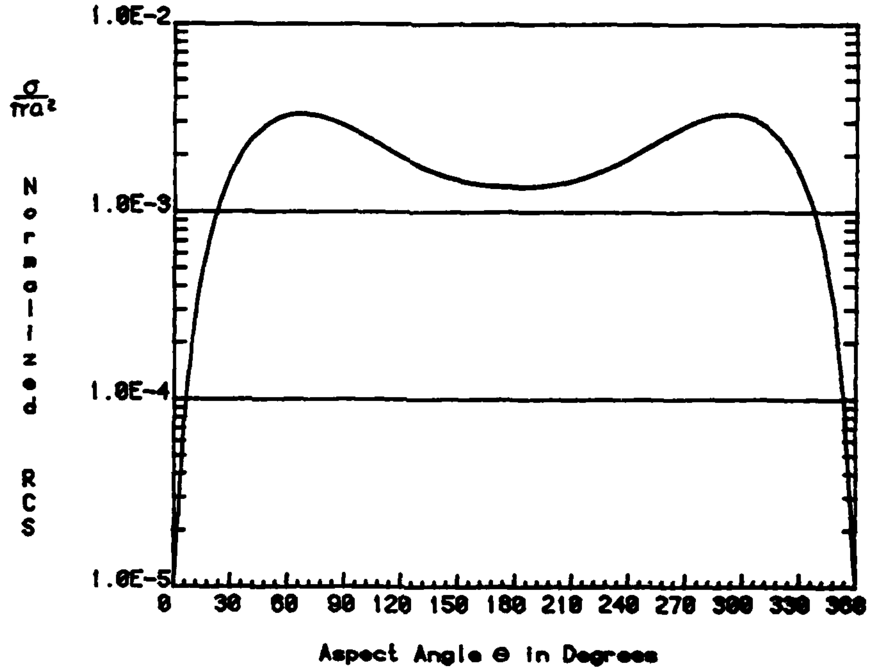


FIGURE 3 NEARFIELD RESULTS FOR A CONE

Frequency: 1 MHz

Observer Radius: 10 M.



Frequency: 1 MHz

Observer Radius: 50 M.

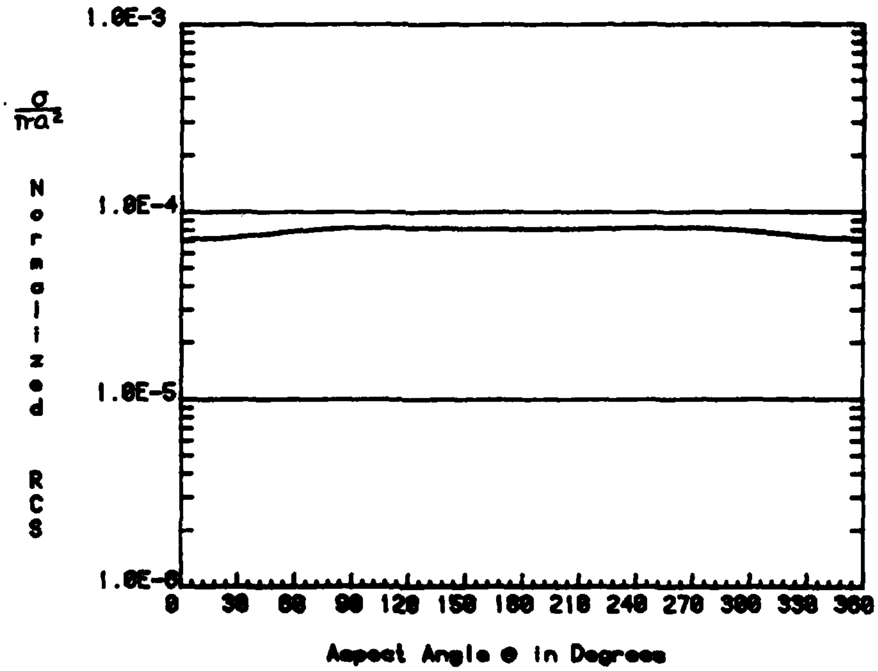


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)

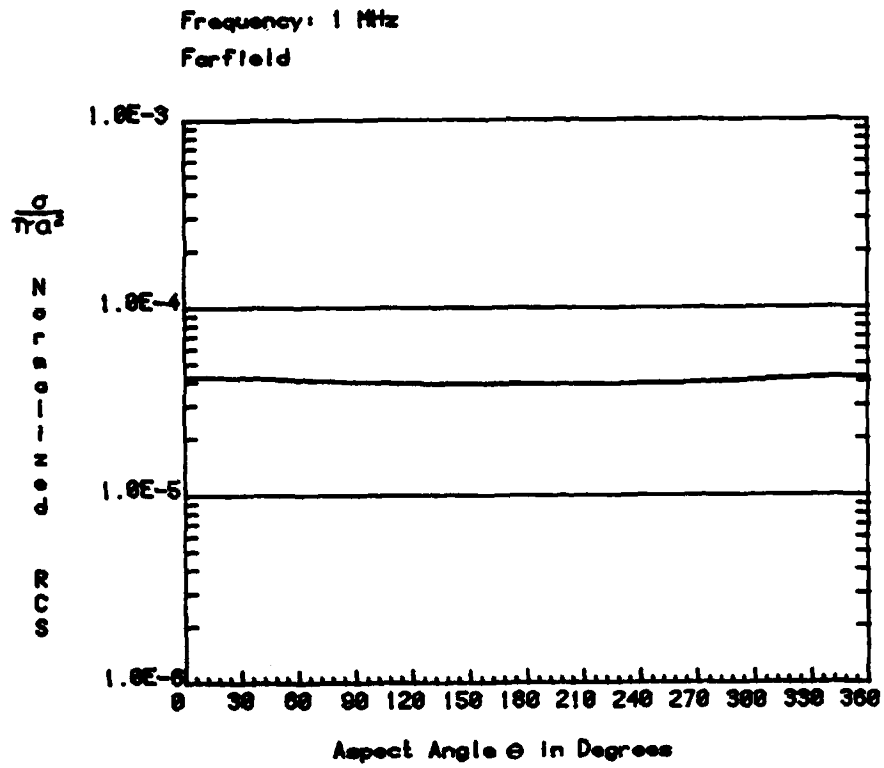
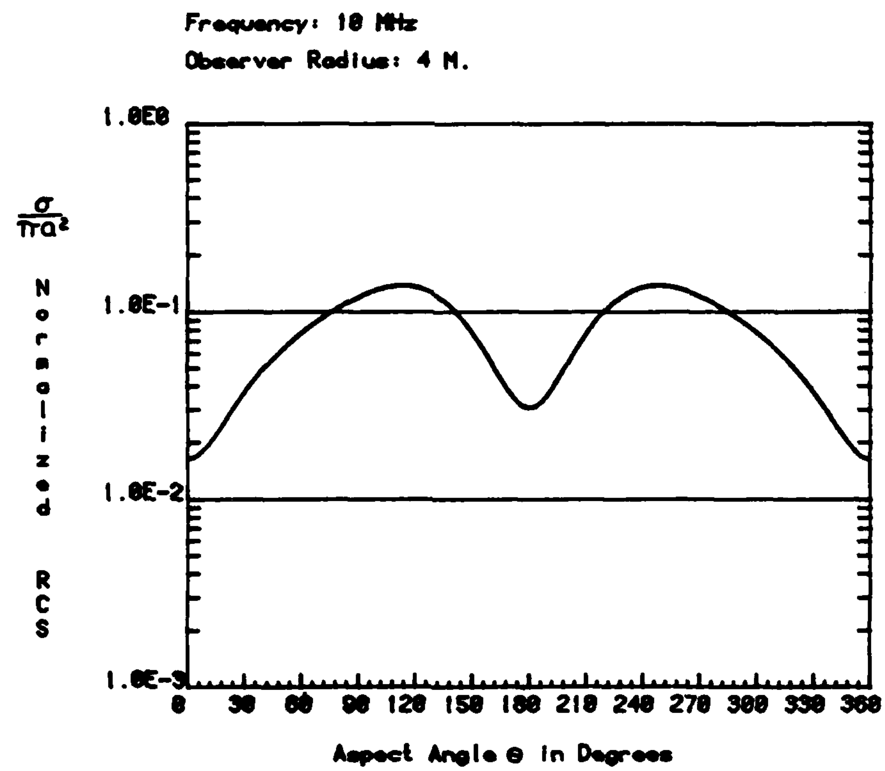


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)



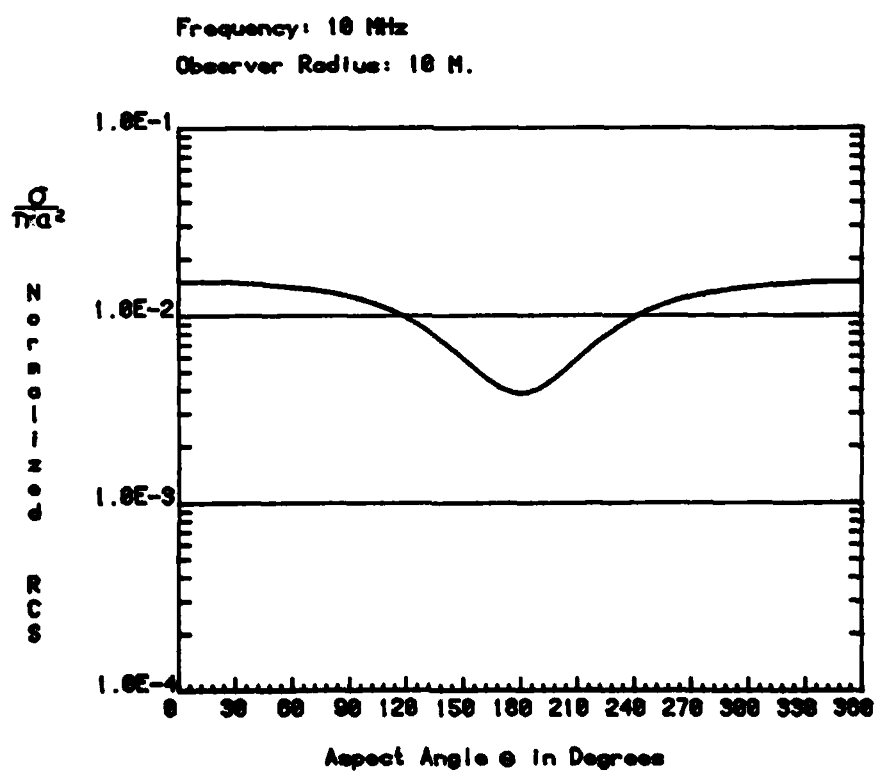
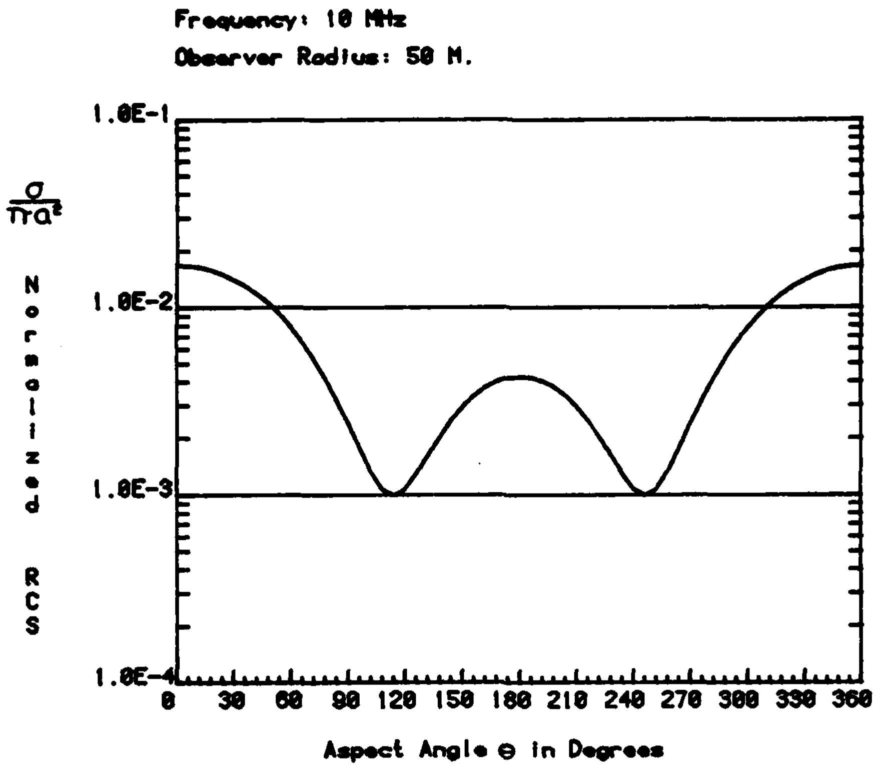


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)



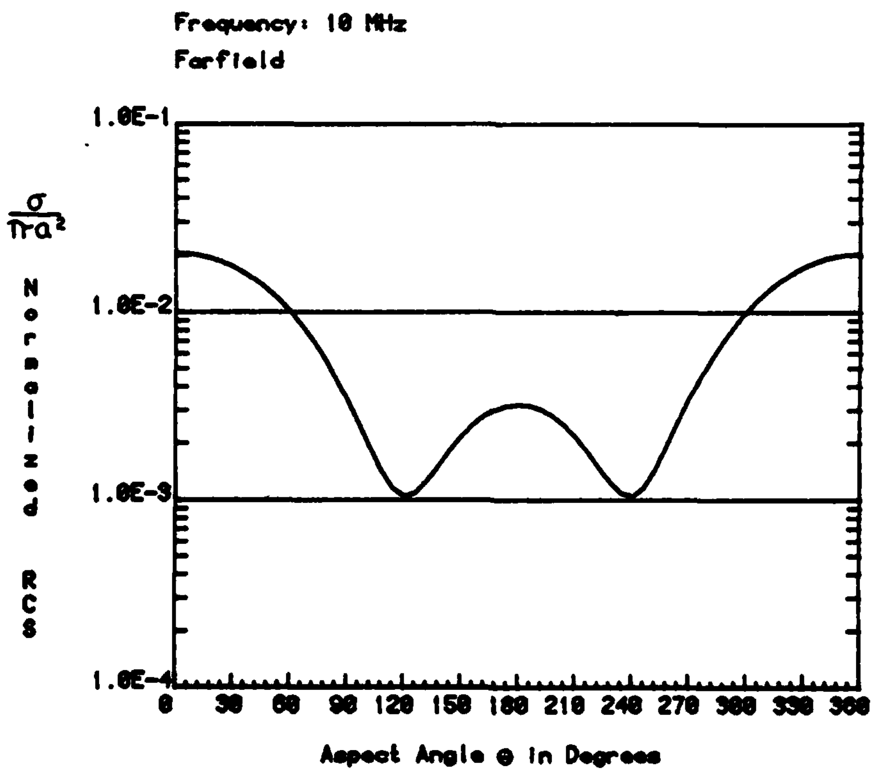
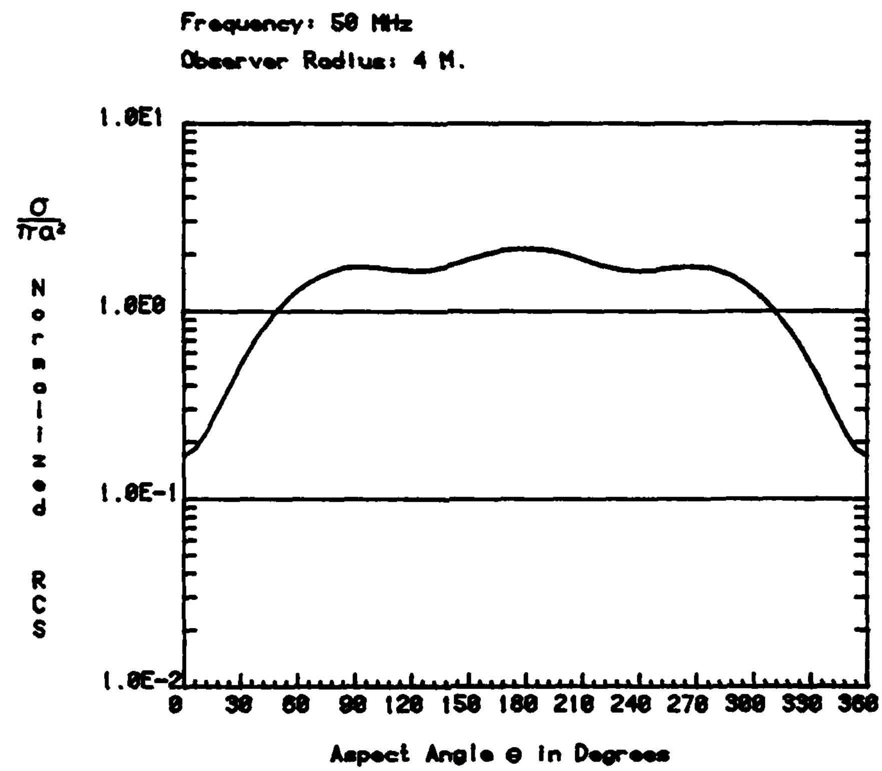


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)



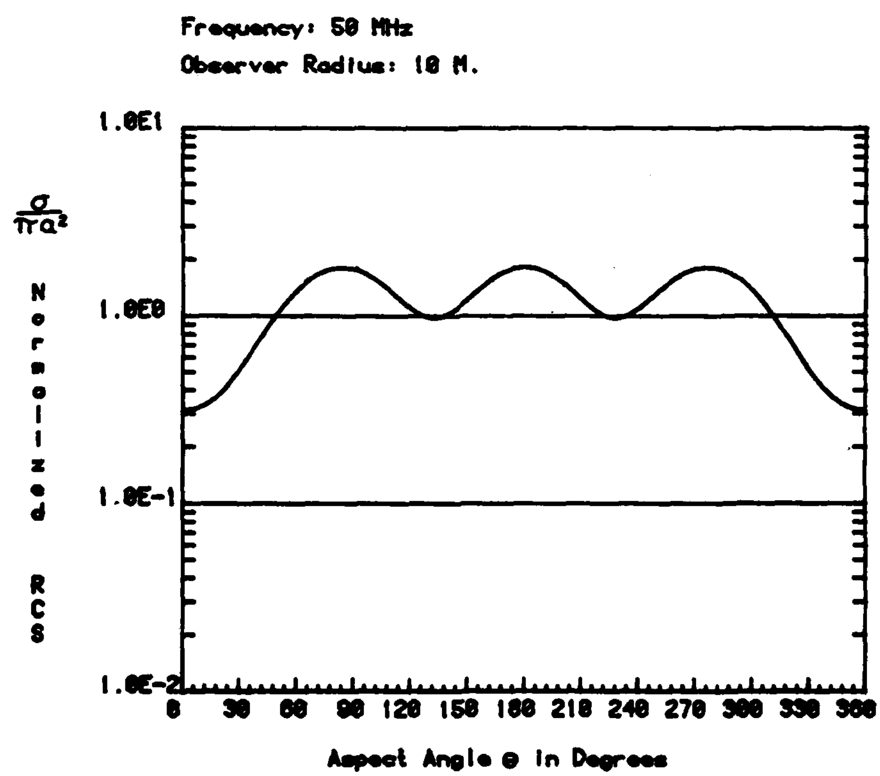
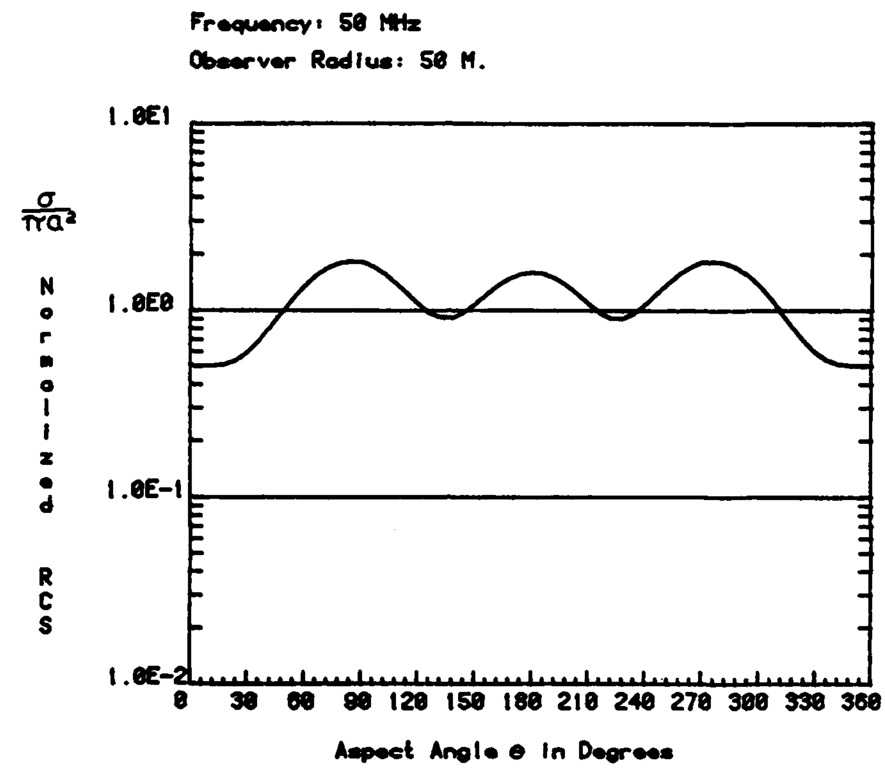


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)



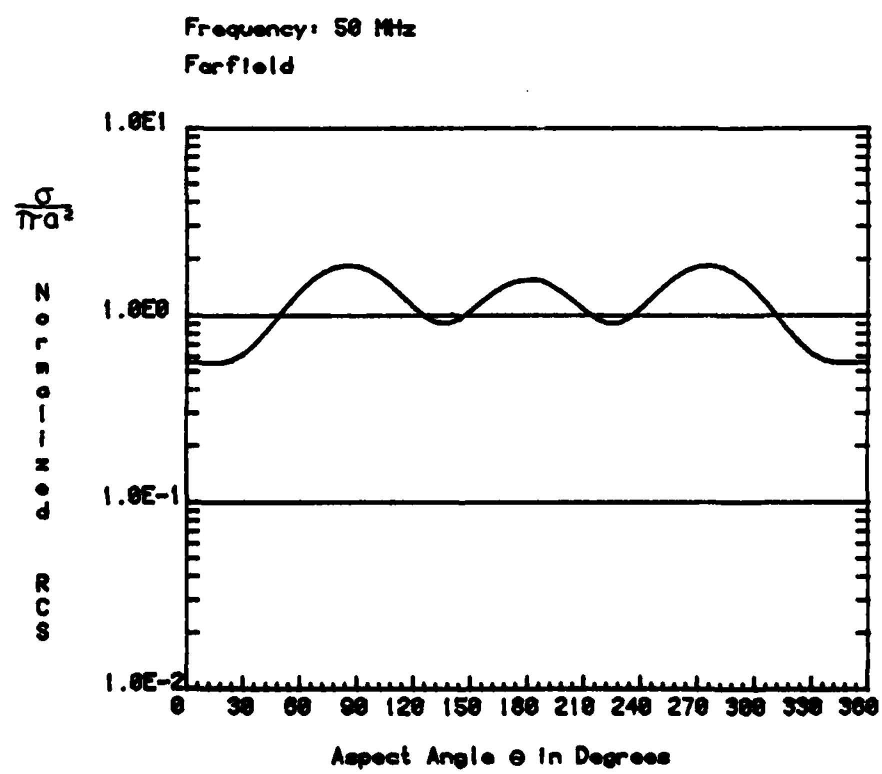


FIGURE 3 NEARFIELD RESULTS FOR A CONE. (CONT.)

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